The University of Texas at Austin
Dept. of Electrical and Computer Engineering
Final Exam ***Solutions 1.0***

Date: December 11, 2023 Course: EE 313 Evans

Name: ____
Last, First

- This in-person exam is scheduled to last two hours.
- Open books, open notes, and open class materials, including homework assignments and solution sets and previous midterm exams and solutions.
- Calculators are allowed.
- You may use any standalone computer system, i.e. one that is not connected to a network.
- ***Please disable all wireless connections on your calculator(s) and computer system(s).***
- Please mute all computer systems.
- Please turn off all phones.
- No headphones are allowed.
- All work should be performed on the midterm exam. If more space is needed, then use the backs of the pages.
- **Fully justify your answers.** If you decide to quote information from a source, please give the quote, page number and source citation**.**

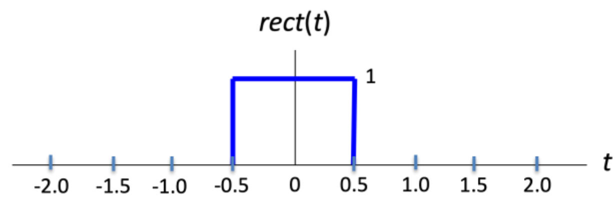
Problem	**Point Value**	**Your Score**	**Topic**
1	16		Continuous-Time Rectangular Pulse
2	18		Discrete-Time Convolution
3	18		Continuous-Time System Properties
4	18		Discrete-Time Filter Design
5	16		Continuous-Time Downconversion
6	14		Discrete-Time Mystery Systems
Total	100		

Problem 1. Continuous-Time Rectangular Pulse. *16 points*

Consider a rectangular pulse signal $rect(t)$ defined as

$$rect(t) = \begin{cases} 1 & \text{for} -0.5 \le t < 0.5 \\ 0 & \text{otherwise} \end{cases}$$

which is plotted on the right.

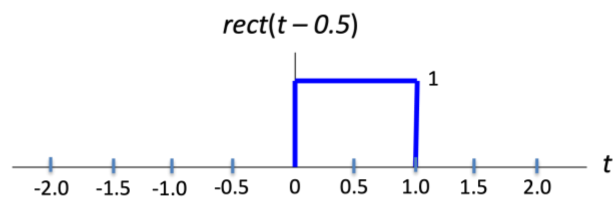


(a) Plot $rect(t - 0.5)$. *4 points*.

This delays $\boldsymbol{rect(t)}$ by 0.5 seconds; i.e., $\boldsymbol{rect(t)}$ is shifted to the right by 0.5 seconds.

$$\boldsymbol{rect(t - 0.5) = \begin{cases} 1 & \textbf{for} -0.5 \le t - 0.5 < 0.5 \\ 0 & \textbf{otherwise} \end{cases}}$$

$$\boldsymbol{rect(t - 0.5) = \begin{cases} 1 & \textbf{for } 0 \le t < 1.0 \\ 0 & \textbf{otherwise} \end{cases}}$$



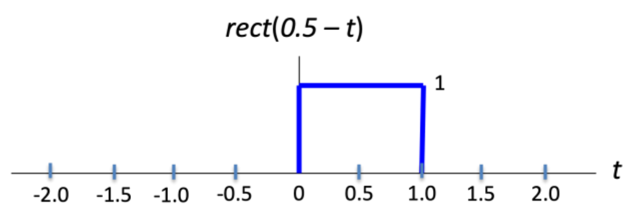
(b) Plot $rect(0.5 - t)$. *4 points*.

This flips $\boldsymbol{rect(t)}$ in time and then delays it by 0.5 seconds.

$$\boldsymbol{rect(0.5 - t) = \begin{cases} 1 & \textbf{for} -0.5 \le 0.5 - t < 0.5 \\ 0 & \textbf{otherwise} \end{cases}}$$

$$\boldsymbol{rect(0.5 - t) = \begin{cases} 1 & \textbf{for} -1.0 \le -t < 0.0 \\ 0 & \textbf{otherwise} \end{cases}}$$

$$\boldsymbol{rect(0.5 - t) = \begin{cases} 1 & \textbf{for } 0.0 < t \le 1.0 \\ 0 & \textbf{otherwise} \end{cases}}$$

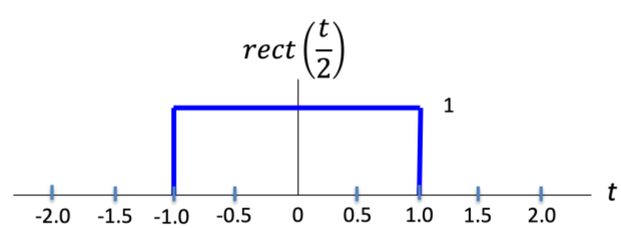


(c) Plot $rect\left(\frac{t}{2}\right)$. *4 points.*

Rectangular pulse will double in width.

$$\boldsymbol{rect\left(\frac{t}{2}\right) = \begin{cases} 1 & \textbf{for} -0.5 \le \frac{t}{2} < 0.5 \\ 0 & \textbf{otherwise} \end{cases}}$$

$$\boldsymbol{rect\left(\frac{t}{2}\right) = \begin{cases} 1 & \textbf{for} -1.0 \le t < 1.0 \\ 0 & \textbf{otherwise} \end{cases}}$$



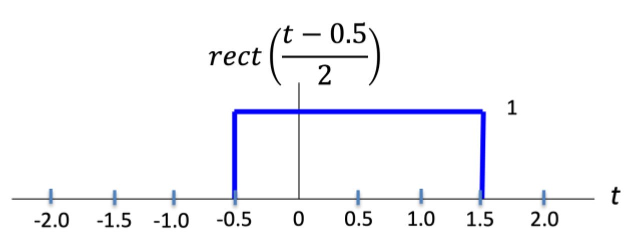
(d) Plot $rect\left(\frac{t-0.5}{2}\right)$. *4 points.*

Rectangular pulse will be doubled in width and then delayed by 0.5 seconds.

$$\boldsymbol{rect\left(\frac{t - 0.5}{2}\right) = \begin{cases} 1 & \textbf{for} -0.5 \le \frac{t - 0.5}{2} < 0.5 \\ 0 & \textbf{otherwise} \end{cases}}$$

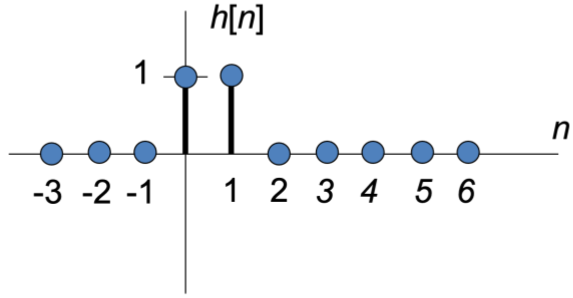
$$\boldsymbol{rect\left(\frac{t - 0.5}{2}\right) = \begin{cases} 1 & \textbf{for} -1.0 \le t - 0.5 < 1.0 \\ 0 & \textbf{otherwise} \end{cases}}$$

$$\boldsymbol{rect\left(\frac{t - 0.5}{2}\right) = \begin{cases} 1 & \textbf{for} -0.5 \le t < 1.5 \\ 0 & \textbf{otherwise} \end{cases}}$$



Problem 2. Discrete-Time Convolution. *18 points*

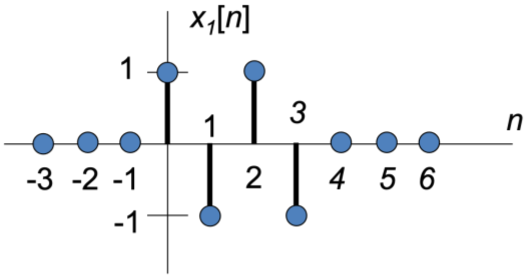
Consider a discrete-time linear time-invariant (LTI) system with impulse response $h[n] = \delta[n] + \delta[n-1]$ plotted on the right.



For each of the following input signals,
i. give a formula for the input signal. *2 points each.*
ii. plot the output signal $y[n]$. *4 points each.*

(a) $\boldsymbol{x_1[n] = \delta[n] - \delta[n-1] + \delta[n-2] - \delta[n-3]}$

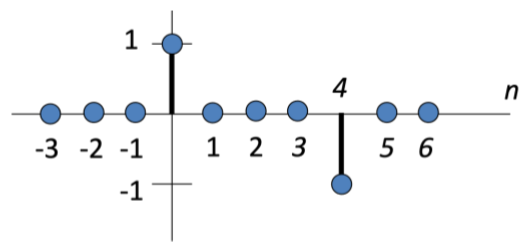
Here, $x_1[n]$ has four non-zero values.



Plot $y[n] = h[n] * x_1[n]$

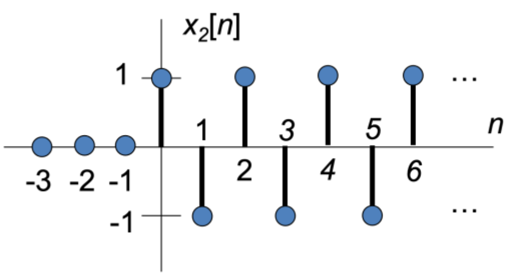
conv([1 1], [1, -1, 1, -1])

$\boldsymbol{y[n] = \delta[n] - \delta[n-4]}$



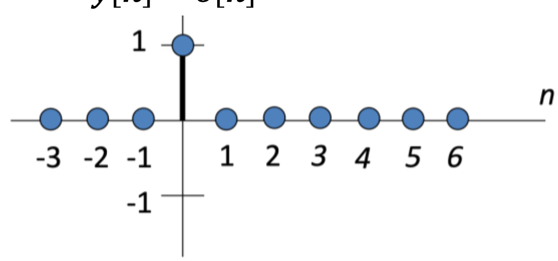
(b) $\boldsymbol{x_2[n] = (-1)^n\, u[n]}$

Here, $x_2[n]$ is 0 for $n < 0$. For $n \geq 0$, $x_2[n]$ alternates between 1 and -1 indefinitely.



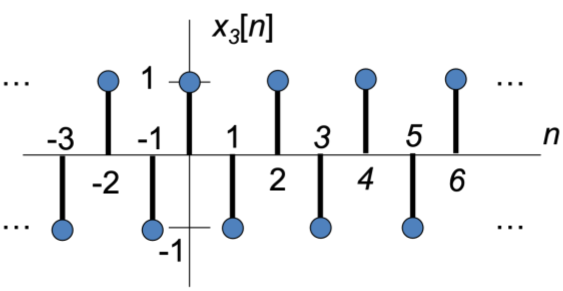
Plot $y[n] = h[n] * x_2[n]$

$\boldsymbol{y[n] = \delta[n]}$



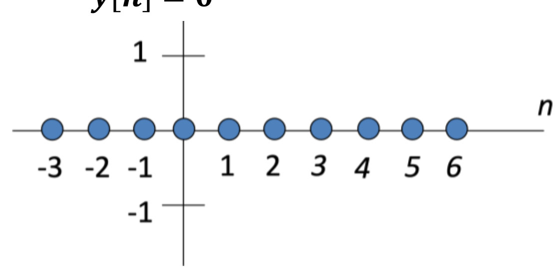
(c) $\boldsymbol{x_3[n] = (-1)^n}$

Here, $x_3[n]$ alternates between 1 and -1 for all n.



Plot $y[n] = h[n] * x_3[n]$

$\boldsymbol{y[n] = 0}$



Problem 3. Continuous-Time System Properties. *18 points*

Each continuous-time system has input $x(t)$ and output $y(t)$, and $x(t)$ and $y(t)$ might be complex-valued.

Determine if each system is linear or nonlinear, time-invariant or time-varying, and bounded-input bounded-output (BIBO) stable or unstable.

You must either prove that the system property holds in the case of linearity, time-invariance, or stability, or provide a counter-example that the property does not hold. Providing an answer without any justification will earn 0 points.

Part	*System Name*	*System Formula*	*Linear?*	*Time-Invariant?*	*BIBO Stable?*
(a)	Integrator	$y(t) = C_0 + \int_{0^-}^{t} x(u)\, du$ for $t \geq 0^-$ and $C_0 = 5$	**NO**	**NO**	**NO**
(b)	Amplitude Modulation	$y(t) = x(t) \cos(2\, \pi\, f_c\, t)$ for $t \geq 0$ where f_c is a constant	**YES**	**NO**	**YES**
(c)	Reciprocal	$y(t) = \frac{1}{x(t)}$ for $-\infty < t < \infty$	**NO**	**YES**	**NO**

***Linearity*. We'll first apply the all-zero input test. If the output is not zero for all time, then the system is not linear. Otherwise, we'll have to apply the definitions for homogeneity and additivity. All-zero input test is a special case of homogeneity $a\, x(t) \rightarrow a\, y(t)$ when the constant $a = 0$.**

***BIBO Stability*. Bounded input $|\, x(t)| \leq B < \infty$ would give bounded output $|y(t)| \leq C < \infty$.**

(a) Integrator: $y(t) = C_0 + \int_{0^-}^{t} x(u)\, du$ for $t \geq 0^-$ and $C_0 = 5$. *6 points.*

***Linearity*. When $x(t) = 0$ for $t \geq 0^-$, $y(t) = C_0 = 5$ for $t \geq 0^-$. Fails all-zero input test. NO.**

***Time-Invariance*. $y_{shifted}(t) = y(t - t_0)$? $y(t - t_0) = C_0 + \int_{0^-}^{t-t_0} x(u)\, du$ for $t \geq 0^-$ and $C_0 = 5$. Input $x(t - t_0)$ and output is $y_{shifted}(t) = C_0 + \int_{0^-}^{t} x(u - t_0)\, du$. Per the class handout "Time Invariance for an Integrator", $C_0 = 0$ is a necessary condition for time-invariance to hold. NO.**

***BIBO Stability*. Let $x(t) = u(t)$. $y(t) = C_0 + \int_{0^-}^{t} u(\lambda)\, d\lambda$ which grows without bound. NO.**

(b) Amplitude Modulation: $y(t) = x(t) \cos(2\, \pi\, f_c\, t)$ for $t \geq 0$ where f_c is a constant. *6 points.*

***Linearity*. Passes all-zero input test. Satisfies homogeneity and additivity properties (below). YES.**

Homogeneity. Input $a\, x(t)$. $y_{scaled}(t) = \left(a\, x(t)\right) \cos(2\, \pi\, f_c\, t) = a\left(x(t) \cos(2\, \pi\, f_c\, t)\right) = a\, y(t)$. YES.

Additivity. Input $x_1(t) + x_2(t)$. $y_{additive}(t) = \left(x_1(t) + x_2(t)\right) \cos(2\, \pi\, f_c\, t) = x_1(t) \cos(2\, \pi\, f_c\, t) + x_2(t) \cos(2\, \pi\, f_c\, t) = y_1(t) + y_2(t)$. YES.

***Time-Invariance*. $y_{shifted}(t) = y(t - t_0)$? $y(t - t_0) = x(t - t_0) \cos\left(2\, \pi\, f_c\, (t - t_0)\right)$. Input $x(t - t_0)$ and output is $y_{shifted}(t) = x(t - t_0) \cos(2\, \pi\, f_c\, t)$. NO.**

***BIBO Stability*. $|y(t)| = |x(t) \cos(2\, \pi\, f_c\, t)| = |x(t)|\, |\cos(2\, \pi\, f_c\, t)| \leq |x(t)| < B < \infty$. YES.**

(c) Reciprocal: $y(t) = \frac{1}{x(t)}$ for $-\infty < t < \infty$. *6 points.*

***Linearity*: Does not pass the all-zero input test. When $x[n] = 0$ for $-\infty < n < \infty$, $y[n] = \frac{1}{0}$. If we take the limit as $x[n] \rightarrow 0$, then $y[n] \rightarrow \infty$. NO.**

***Time-Invariance*: Pointwise operation; current output value $y[n]$ depends only on current input $x[n]$ and not on any other input/output values. All pointwise operations are time-invariant. YES.**

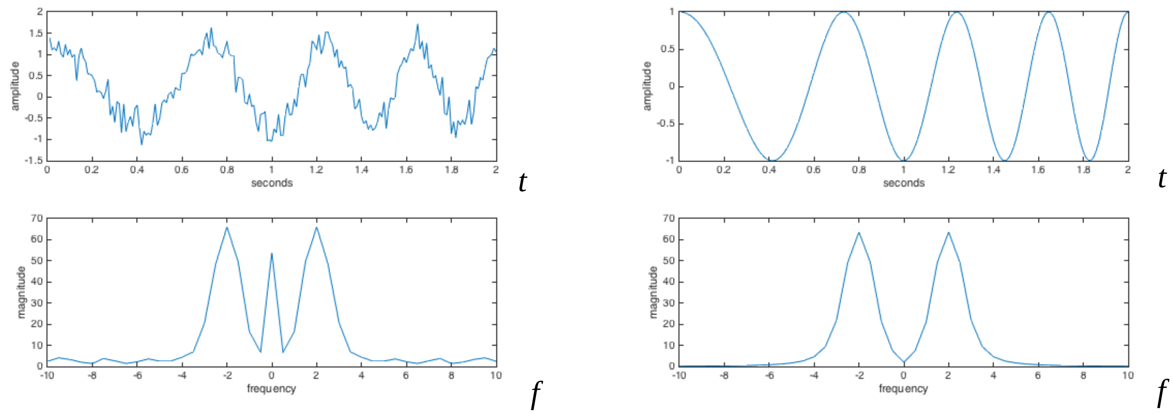
***BIBO Stability*. When $x[n] = 0$ for $-\infty < n < \infty$, $y[n] \rightarrow \infty$ in the limit. No bounded. NO.**

Problem 4. Discrete-Time Filter Design. *18 points*.

A sinusoidal signal of interest has a principal frequency that can vary over time in the range 1-3 Hz. Using a sampling rate of f_s = 20 Hz, a sinusoidal signal was acquired for 2s and shown below on the left in the upper plot. The lower plot is the magnitude of the signal's frequency content.

The acquired signal has interference and other impairments that reduce the signal quality.

The signal shown below on the right is the sinusoidal signal without the impairments.



Design a second-order infinite impulse response (IIR) filter to filter the acquired signal above on the left to give the sinusoidal signal above on the right. Filter should be bounded-input bounded-output stable.

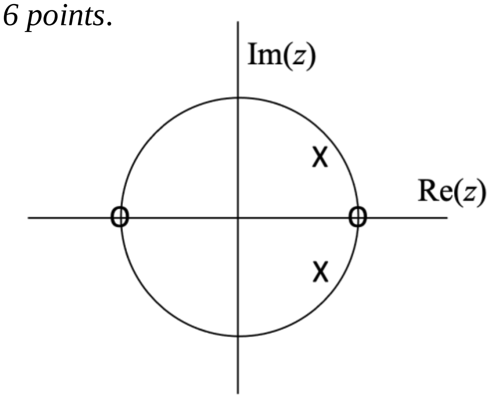
(a) Give the poles and zeros of the second-order IIR filter. Explain why you chose the poles and zeros. *12 points*.

***Passband*: 1-3 Hz with center frequency f_c of 2 Hz. Two poles would be at angles equal to the center frequency and its negative counterpart, with magnitudes close to but inside the unit circle: $0.9\, e^{j\,\hat{\omega}_c}$ and $0.9\, e^{-j\,\hat{\omega}_c}$. $\hat{\omega}_c$ is computed on right.**

$$\hat{\omega}_c = 2\pi \frac{f_c}{f_s} = 2\pi \frac{2\ Hz}{20\ Hz} = \frac{\pi}{5}\ \text{rad/sample}$$

***Stopbands*: From the above magnitude plot of the left, impairments are around DC (0 Hz) as well as 4-10 Hz and the negative counterparts -10 Hz to -4 Hz. Two zeros on unit circle at angles corresponding to the center frequencies of the two stopbands, i.e., 0 Hz and 10 Hz which are 0 and $2\,\pi\, \frac{10\ Hz}{20\ Hz} = \pi$ rad/sample: $e^{j\,0} = 1$ and $e^{j\,\pi} = -1$.**

(b) Draw the pole-zero diagram for the second-order IIR filter. *6 points*.



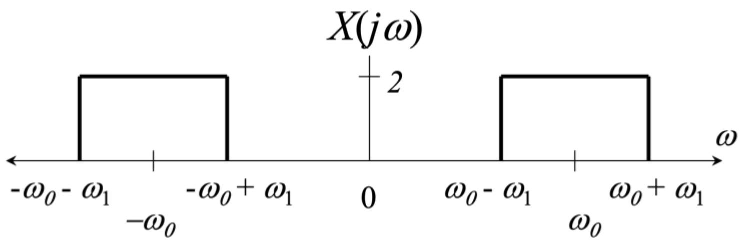
Problem 5. Continuous-Time Downconversion. *16 points*.

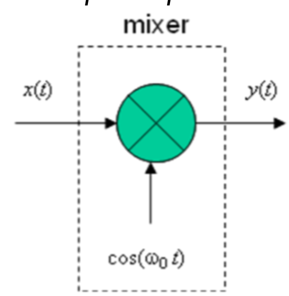
A signal $x(t)$ is input to a mixer to produce the output $y(t)$ where

$$y(t) = x(t)\cos(\omega_0\, t)$$

where $\omega_0 = 2\,\pi\, f_0$ and $f_0 = 5$ kHz. A block diagram of the mixer is shown below on the right.

The Fourier transform of $x(t)$ is shown below on the left where $\omega_1 = 2\,\pi\, f_1$ and $f_1 = 1$ kHz.





(a) Using Fourier transform properties, derive an expression for $Y(j\omega)$ in terms of $X(j\omega)$. *8 points.*

Approach #1: $y(t) = x(t)\ \cos(w_0 t)$

$$Y(jw) = \frac{1}{2p} X(jw) * \left(pd(w+w_0) + pd(w-w_0)\right)$$

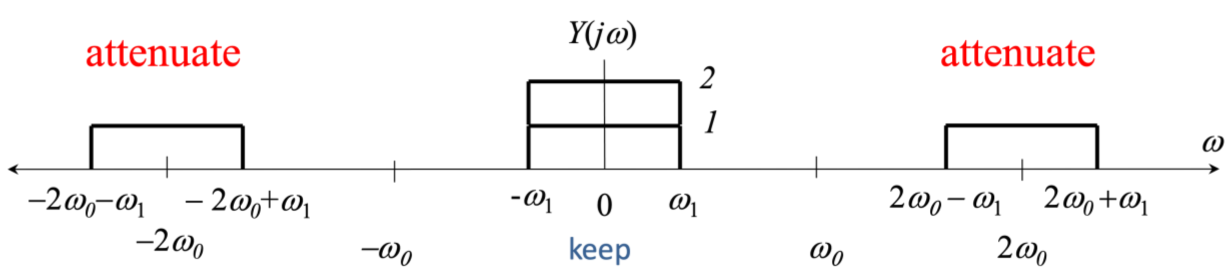
Recall that $x(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} \delta(\tau - t_0)\, x(t-\tau)\, d\tau = x(t-t_0)$

$$Y(jw) = \frac{1}{2} X\left(j(w+w_0)\right) + \frac{1}{2} X\left(j(w-w_0)\right)$$

Approach #2: $\boldsymbol{x(t)\ \cos(\omega_0\, t) = x(t)\ \left(\frac{1}{2}\, e^{j\,\omega_0\, t} + \frac{1}{2} e^{-j\,\omega_0\, t}\right)}$

Apply the frequency shift property for the continuous-time Fourier transform per *Signal Processing First*, Sec. 12.2-1.

(b) Sketch $Y(j\omega)$ vs. ω. Label all important points on the horizontal and vertical axes. *6 points.*



(c) What operation would you apply to the signal $y(t)$ in part (b) to pass frequencies from $-\omega_1$ to ω_1 and attenuate other frequencies? This will allow us to recover the message signal that was transmitted using amplitude modulation. *2 points.*

Lowpass filter that would pass frequencies from $-\omega_1$ to ω_1 and attenuate frequencies from $-2\omega_0 - \omega_1$ to $-2\omega_0 + \omega_1$ and from $2\omega_0 - \omega_1$ to $2\omega_0 + \omega_1$.

Problem 6. Discrete-Time Mystery Systems. *14 points*.

You're trying to identify unknown discrete-time systems.

You input a discrete-time chirp signal $x[n]$ and look at the output to figure out what the system is.

The discrete-time chirp is formed by sampling a chirp signal that sweeps 0 to 8000 Hz over 0 to 5s

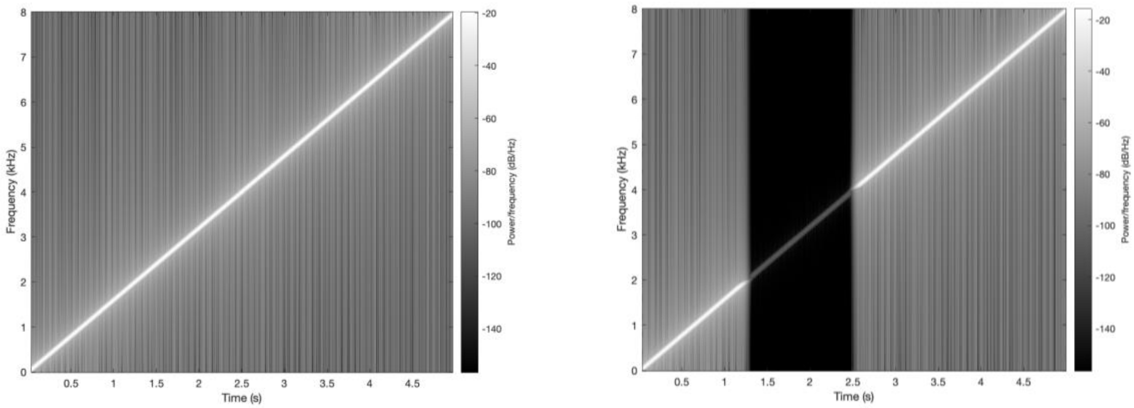
$$x(t) = \cos(2\pi f_1 t + 2\pi \mu t^2)$$

where $f_1 = 0$ Hz, $f_2 = 8000$ Hz, and $\mu = \frac{f_2 - f_1}{2\, t_{\max}} = \frac{8000\text{ Hz}}{10\, s} = 800\text{ Hz}^2$. Sampling rate f_s is 16000 Hz.

In part (a) and (b) blow, identify the unknown system as one of the following **with justification**:

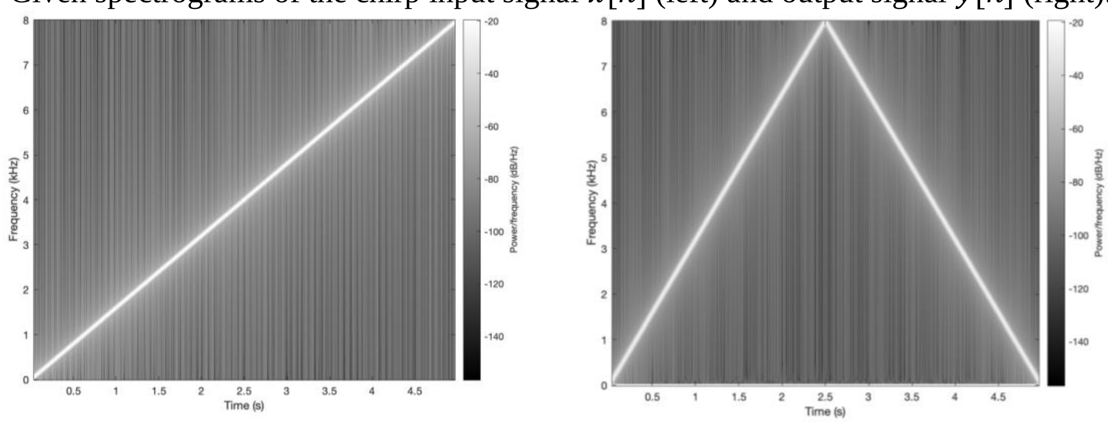
1. filter – give selectivity (lowpass, highpass, bandpass, bandstop) and passband/stopband frequencies
2. pointwise nonlinearity – give the integer exponent k to produce the output $y[n] = x^k[n]$

(a) Given spectrograms of the chirp input signal $x[n]$ (left) and output signal $y[n]$ (right). *7 points*.



In the output spectrogram, principal frequencies of the chirp input signal between 2 and 4 kHz are severely attenuated and other principal frequencies are passed. No new frequencies are created, so it is likely an LTI filter. Bandstop filter. **See next page for the Matlab code.**

(b) Given spectrograms of the chirp input signal $x[n]$ (left) and output signal $y[n]$ (right). *7 points.*



Please note that the output spectrogram has a strong component at DC (0 rad/sample).

The output spectrogram had a strong DC component over all time whereas the input signal only has a DC component at the very beginning of the chirp signal. New frequencies are being created, so it's not an LTI system. At any point in time from 0s to 2.5s, the frequency

component in the output spectrogram that rises from 0 Hz to 8000 Hz is twice the principal frequency component in the input spectrogram. The system is a squaring block. When inputting $\cos(\omega_0\, t)$ into a square block, the output is $\cos^2(\omega_0\, t) = \frac{1}{2} + \frac{1}{2}\cos(2\, \omega_0\, t)$.

```
%% Midterm Problem 2.4
fs = 16000;
Ts = 1 / fs;
tmax = 5;
t = 0 : Ts : tmax;

%% Create chirp signal
f1 = 0;
f2 = fs/2;
mu = (f2 - f1) / (2*tmax);
x = cos(2*pi*f1*t + 2*pi*mu*(t.^2));

%% (a) bandstop filter
fnyquist = fs/2;
fstop1 = 1800;
fpass1 = 2000;
fpass2 = 4000;
fstop2 = 4200;
ctfrequencies = [0 fstop1 fpass1 fpass2 fstop2 fnyquist];
idealAmplitudes = [1 1 0 0 1 1];
pmfrequencies = ctfrequencies / fnyquist;
filterOrder = 400;
h = firpm( filterOrder, pmfrequencies, idealAmplitudes );
h = h / sum(h .^ 2);

y = conv(x, h);

%%% Spectrogram parameters
blockSize = 1024;
overlap = 1023;

%%% Plot spectrogram of input signal
figure;
spectrogram(x, blockSize, overlap, blockSize, fs, 'yaxis');
colormap gray;

%%% Plot spectrogram of output signal
figure;
spectrogram(y, blockSize, overlap, blockSize, fs, 'yaxis');
colormap gray;


%% (b) squaring block
figure;
spectrogram(x, blockSize, overlap, blockSize, fs, 'yaxis');
colormap gray;

%%% Plot spectrogram of output signal
figure;
spectrogram(x.^2, blockSize, overlap, blockSize, fs, 'yaxis');
colormap gray;
```